SOME REPRESENTATIONS IN GENERAL FORM OF SOLUTIONS TO THE EQUATIONS OF THE THEORY OF SHALLOW SHELLS

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1. In a number of cases in the solution of problems of the linear theory of shells, auxiliary functions are introduced which enable the problem to be described by a single equation of high order. The general method by which these functions are introduced reduces to the following [1,2].

Let us suppose that we are given a set of equations in partial derivatives

$$\sum_{i=1}^{n} \cdot a_{ij}\left(\frac{\partial}{\partial x_p}\right) u_j = 0 \qquad (i = 1, \ldots, n;\ 1 \leqslant p \leqslant k) \tag{1.1}$$

where u_j are the unknown functions, $a_{ij}(\partial/\partial x_p)$ are linear differential operators of finite order with constant coefficients. Consider the set of algebraic equations

$$\sum_{i=1}^{n} a_{ij}(\alpha_p) v_j = 0 \qquad (i = 1, \ldots, n;\ 1 \leqslant p \leqslant k) \tag{1.2}$$

derived from (1.1) by a formal replacement of the operation of differentiation with respect to x_p by multiplication by some parameter a_p. Let us suppose that the first $n-1$ equations of (1.2) enable us to express $v_1, \ldots, v_{n-1}$ in terms of v_n

$$v_i = \frac{D_i(\alpha_1, \ldots, \alpha_k)}{R(\alpha_1, \ldots, \alpha_k)} v_n \qquad (i = 1, \ldots, n-1) \tag{1.3}$$

Obviously D_i and R can be taken as polynomials in $a_1, \ldots, a_k$. We introduce the function Φ assuming that

$$u_i = D_i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)\Phi \quad (i = 1, \ldots, n-1), \qquad u_n = R\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)\Phi \tag{1.4}$$

and that Φ satisfies the equation

$$\left\{\sum_{j=1}^{n-1} a_{nj}\left(\frac{\partial}{\partial x_p}\right) D_j\left(\frac{\partial}{\partial x_p}\right) + a_{nn}\left(\frac{\partial}{\partial x_p}\right) R\left(\frac{\partial}{\partial x_p}\right)\right\}\Phi = K\left(\frac{\partial}{\partial x_p}\right)\Phi = 0 \tag{1.5}$$

Here $K(a_p)$ is the determinant of the set of equations (1.2).

It will readily be seen that any function Φ satisfying (1.5) provides, by means of the relations (1.4), a solution to the set of equations (1.1). If the last of Equations (1.1) is nonhomogeneous, then Equation (1.5) will also be nonhomogeneous. We shall apply these ideas to the equilibrium equations of a shallow shell in terms of displacements

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right)u + \frac{1+\nu}{2}\frac{\partial^2}{\partial x \partial y}v - \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\frac{\partial w}{\partial x} = 0 \tag{1.6}$$

$$\frac{1+\nu}{2}\frac{\partial^2}{\partial x \partial y}u + \left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right)v - \left(\frac{1}{R_2} + \frac{\nu}{R_1}\right)\frac{\partial w}{\partial y} = 0 \tag{1.7}$$

$$\nabla^4 w + \frac{B}{D}\left(\frac{1}{R_2^2} + \frac{1}{R_1^2} + \frac{2\nu}{R_1 R_2}\right)w - \frac{B}{D}\left[\frac{1}{R_1}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + \frac{1}{R_2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right)\right] = \frac{q}{D} \tag{1.8}$$

We then have the representation

$$u = -\frac{1+\nu}{2}\left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\frac{\partial^3\Phi}{\partial x \partial y^2} + \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\frac{\partial^3\Phi}{\partial x \partial y^2} + \frac{1-\nu}{2}\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\frac{\partial^3\Phi}{\partial x^3} \tag{1.9}$$

$$v = -\frac{1+\nu}{2}\left(\frac{\nu}{R_2} + \frac{1}{R_1}\right)\frac{\partial^3\Phi}{\partial x^2 \partial y} + \left(\frac{1}{R_2} + \frac{\nu}{R_1}\right)\frac{\partial^3\Phi}{\partial x^2 \partial y} + \frac{1-\nu}{2}\left(\frac{1}{R_2} + \frac{\nu}{R_1}\right)\frac{\partial^3\Phi}{\partial y^3} \tag{1.10}$$

$$w = \frac{1-\nu}{2}\Delta^4\Phi \tag{1.11}$$

The function Φ must satisfy the equation

$$\Delta^8\Phi + \frac{B(1-\nu^2)}{D}\left(\frac{1}{R_2^2}\frac{\partial^4}{\partial x^4} + \frac{2}{R_1 R_2}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{R_1^2}\frac{\partial^4}{\partial y^4}\right)\Phi - \frac{q}{D} = 0 \tag{1.12}$$

In [3], the following set of equations was derived from a consideration of the equilibrium of a shallow shell:

$$\frac{1}{Eh}\nabla^4\varphi - \nabla_k^2 w = 0, \qquad \nabla_k^2\varphi + D\nabla^4 w - Z = 0 \tag{1.13}$$

$$\left(\nabla_k^2 = k_y\frac{\partial^2}{\partial x^2} + k_x\frac{\partial^2}{\partial y^2} - 2k_{xy}\frac{\partial^2}{\partial x \partial y}\right) \tag{1.14}$$

By introducing a function Φ given by the formulas

$$w=\nabla^4\Phi,\qquad \varphi=Eh\nabla_k{}^2\Phi \tag{1.15}$$

we can reduce this set to the one equation

$$\nabla^8\Phi+\frac{12(1-\nu^2)}{h^2}\nabla_k{}^2\nabla k^2\Phi-\frac{Z}{D}=0 \tag{1.16}$$

In [3], an investigation of the equilibrium of a cylindrical shell let to the following set of equations in the displacements (for simplicity we shall assume that only transverse loading is applied):

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right)u+\frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\partial y}+\nu\frac{\partial w}{\partial x}-c^2\left(\frac{\partial^3}{\partial x^3}-\frac{1-\nu}{2}\frac{\partial^3}{\partial x\partial y^2}\right)w=0 \tag{1.17}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\partial y}+\left(\frac{\partial^2}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right)v+\frac{\partial w}{\partial y}+\frac{3-\nu}{2}c^2\frac{\partial^3 w}{\partial x^2\partial y}=0 \tag{1.18}$$

$$\nu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-c^2\frac{\partial}{\partial y}\left[\frac{\partial^2}{\partial y^2}-(2-\nu)\frac{\partial^2}{\partial x^2}\right]v+c^2(\nabla^4+1)w+\frac{1-\nu}{Eh}R^2Z=0$$

$$c^2=\frac{h^2}{12R^2} \tag{1.19}$$

Equations (1.17) to (1.19) differ from Equations (1.6) to (1.8) in that they contain a number of additional terms. They can be reduced to a single equation with the aid of the expressions

$$u=c^2\left(\frac{\partial^5}{\partial x^5}-\frac{\partial^5}{\partial x\partial y^4}\right)\Phi+\frac{\partial^3\Phi}{\partial x\partial y^2}-\nu\frac{\partial^3\Phi}{\partial x^3} \tag{1.20}$$

$$v=2c^2\left(\frac{\partial^5}{\partial x^4\partial y}+\frac{\partial^5}{\partial x^2\partial y^3}\right)\Phi-(2+\nu)\frac{\partial^3\Phi}{\partial x^2\partial y}-\frac{\partial^3\Phi}{\partial y^3} \tag{1.21}$$

$$w=\nabla^4\Phi \tag{1.22}$$

and we obtain the following equation for Φ:

$$c^2(\nabla^2+1)^2\nabla^4\Phi-2c^2(1-\nu)\left(\frac{\partial^4}{\partial x^4}-\frac{\partial^4}{\partial x^2\partial y^2}\right)\nabla^2\Phi+(1-\nu)\frac{\partial^4\Phi}{\partial x^4}=\frac{(1-\nu^2)h}{12Ec^2}Z \tag{1.23}$$

Other versions of the theory of shells can also be simplified by the introduction of auxiliary functions.

It appears that Mishonov [4] was the first to point out that Equations (1.15) are not always valid in the case of a spherical shell.

This fact indicates that auxiliary functions should be used with caution and renders pertinent the following questions.

1) For what types of shells does the introduction of auxiliary functions with the aid of (1.9) to (1.11), (1.15) enable us to investigate any type of state of stress?

2) If for certain types of shells Equations (1.9) to (1.11), (1.15)

cannot always be used, then what states of stress of these shells do these equations describe?

3) Can any state of stress for a cylindrical shell be described by Equations (1.20) to (1.22)?

4) What is the degree of arbitrariness of the function Φ in the above representations?

We shall consider these questions in the following sections.

2. Let us consider the possibility of realizing (1.9) to (1.11). Suppose that there exist three arbitrary sufficiently smooth functions u, v, w related by Equations (1.6), (1.7). We shall try to find a function Φ by means of which solutions (1.9) to (1.11) are realized. We shall suppose for simplicity that the region occupied by the plan of the shell is simply-connected. From (1.6) to (1.7) we have

$$\left\{ \begin{matrix} u \\ v \end{matrix} \right\} = K^{-1} \left\{ \begin{matrix} \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right) \frac{\partial w}{\partial x} \\ \left(\frac{1}{R_2} + \frac{\nu}{R_1}\right) \frac{\partial w}{\partial y} \end{matrix} \right\} + \left\{ \begin{matrix} u_0 \\ v_0 \end{matrix} \right\} \tag{2.1}$$

where K is the operator of the plane problem in the theory of elasticity for a region with zero conditions on the boundary, and u_0, v_0 represent some solution to the homogeneous plane problem of the theory of elasticity for Ω. Also, from (1.11) we have

$$\Phi = \frac{2}{1-\nu} Z^{-1} w + \Phi_0 \tag{2.2}$$

where Z is the biharmonic operator for Ω with zero conditions on the boundary, and Φ_0 is biharmonic in the Ω-function.

We introduce now the operator C which assigns to each sufficiently smooth function Φ a pair of functions u, v according to Formulas (1.9), (1.10). Evidently, the biharmonic function Φ_0, as follows from (2.1) and (2.2), must be given by the relations

$$\left\{ \begin{matrix} u \\ v \end{matrix} \right\} = C\Phi = \frac{2}{1-\nu} CZ^{-1} w + C\Phi_0 = K^{-1} \left\{ \begin{matrix} \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right) \frac{\partial w}{\partial x} \\ \left(\frac{1}{R_2} + \frac{\nu}{R_1}\right) \frac{\partial w}{\partial y} \end{matrix} \right\} + \left\{ \begin{matrix} u_0 \\ v_0 \end{matrix} \right\} \tag{2.3}$$

From (2.3) we have

$$C\Phi_0 = K^{-1} \left\{ \begin{matrix} \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right) \frac{\partial w}{\partial x} \\ \left(\frac{1}{R_2} + \frac{\nu}{R_1}\right) \frac{\partial w}{\partial y} \end{matrix} \right\} - \frac{2}{1-\nu} CZ^{-1} w + \left\{ \begin{matrix} u_0 \\ v_0 \end{matrix} \right\} \tag{2.4}$$

We will show that the right-hand side of Formula (2.4) is a solution of the homogeneous equations in the plane problem of the theory of elasticity for Ω. In order to do so, it is sufficient simply to establish the relation

$$\frac{2}{1-\nu}\, KCZ^{-1}w = \begin{Bmatrix} \left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)\frac{\partial w}{\partial x} \\ \left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)\frac{\partial w}{\partial y} \end{Bmatrix} \tag{2.5}$$

We shall evaluate the operator *KC* by making use of the symbolic method and by denoting the operator of differentiation with respect to *x* by *p*, and the operator of differentiation with respect to *y* by *q*. We then have

$$KC = \begin{Bmatrix} \left(p^2+\frac{1-\nu}{2}\,q^2\right)C_1+\frac{1+\nu}{2}\,pqC_2 \\ \left(q^2+\frac{1-\nu}{2}\,p^2\right)C_2+\frac{1+\nu}{2}\,pqC_1 \end{Bmatrix} \tag{2.6}$$

Here C_1, C_2 are given by (1.9), (1.10):

$$\begin{aligned} C_1 &= -\frac{1+\nu}{2}\left(\frac{\nu}{R_1}+\frac{1}{R_2}\right)pq^2+\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)pq^2+\frac{1-\nu}{2}\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)p^3 \\ C_2 &= -\frac{1+\nu}{2}\left(\frac{\nu}{R_2}+\frac{1}{R_1}\right)p^2q+\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)p^2q+\frac{1-\nu}{2}\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)q^3 \end{aligned} \tag{2.7}$$

From (2.6), (2.7) we obtain

$$KC = \begin{Bmatrix} \frac{1-\nu}{2}\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)p\,(p^2+q^2)^2 \\ \frac{1-\nu}{2}\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)q\,(p^2+q^2)^2 \end{Bmatrix} \tag{2.8}$$

Also, if we take into account that $(p^2+q^2)^2Z^{-1}w \equiv w$ we can easily obtain (2.5) from (2.8). It has thus been established that the right-hand side of (2.4) is a solution to the homogeneous equations of the plane problem of the theory of elasticity. We shall denote this solution by u_1, v_1. Thus the biharmonic function Φ_0 must be determined by the set

$$C\Phi_0 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \tag{2.9}$$

which in expanded form can be written as

$$\begin{aligned} -\frac{1+\nu}{2}\left(\frac{\nu}{R_1}+\frac{1}{R_2}\right)\frac{\partial^3\Phi_0}{\partial x\partial y^2}+\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)\frac{\partial^3\Phi_0}{\partial x\partial y^2}+\frac{1-\nu}{2}\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)\frac{\partial^3\Phi_0}{\partial x^3} &= u_1 \\ -\frac{1+\nu}{2}\left(\frac{\nu}{R_2}+\frac{1}{R_1}\right)\frac{\partial^3\Phi_0}{\partial x^2\partial y}+\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)\frac{\partial^3\Phi_0}{\partial x^2\partial y}+\frac{1-\nu}{2}\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)\frac{\partial^3\Phi_0}{\partial y^3} &= v_1 \end{aligned} \tag{2.10}$$

In order to determine Φ_0 from (2.10), we make use of the Goursat relation $\Phi_0 = \bar{z}a + z\bar{a} + b + \bar{b}$. After substituting this relation in (2.10), multiplying the second of Equations (2.10) by i and adding to the first, we find

$$a''(3-\nu)(1-\nu)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)-$$
$$-(1-\nu^2)\left[(z\overline{a'''}+\overline{b'''})\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+2\overline{a''}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]=u_1+iv_1 \qquad (2.11)$$

Further, on the basis of the well-known Kolosov-Muskhelishvili relation

$$2\mu(u_1+iv_1)=\varkappa\varphi-z\overline{\varphi'}-\bar{\psi} \qquad (2.12)$$

Here ϕ, ψ are analytic in the Ω-function. Thus a, b must be determined from the relation

$$a''(3-\nu)(1-\nu)-(1-\nu^2)\left[(z\overline{a'''}+\overline{b'''})\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+2\overline{a''}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]=$$
$$=\frac{1}{2\mu}(\varkappa\varphi-z\overline{\varphi'}-\overline{\psi}) \qquad (2.13)$$

It follows from (2.12) that

$$a''(3-\nu)(1-\nu)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{\varkappa\varphi}{2\mu}, \qquad a'''(1-\nu^2)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{\varphi'}{2\mu}$$
$$2a''\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+b'''\left(\frac{1}{R_2}-\frac{1}{R_1}\right)=-\frac{\psi}{2\mu(1-\nu^2)} \qquad (2.14)$$

Also, if we take into account that in the case of a plane state of stress $\kappa = (3-\nu)/(1+\nu)$, we see immediately that a and b can always be found from (2.14) if $1/R_1 - 1/R_2 \neq 0$.

Thus, provided the shell is not spherical, solutions (1.9), (1.10) are always realizable. It follows also from (2.14) that if Φ_1, Φ_2 satisfy simultaneously (1.9), (1.10) for given u, v, w, then

$$\Phi_1-\Phi_2=m_1\left[\left(\frac{1}{R_2}-\frac{2+\nu}{R_1}\right)x^3+3\left(\frac{1}{R_1}+\frac{\nu}{R_2}\right)xy^2\right]+$$
$$+m_2\left[\left(\frac{1}{R_1}-\frac{2+\nu}{R_2}\right)y^3+3\left(\frac{1}{R_2}+\frac{\nu}{R_1}\right)x^2y\right]+P \qquad (2.15)$$

where m_1, m_2 are arbitrary constants and P is an arbitrary polynomial of the second order. Consequently, the function Φ for given values of u, v, w can be determined to the accuracy of eight constants.

We shall now consider the case of a spherical shell.

If $R_1 = R_2 = R$, Expressions (1.9), (1.10) give

$$u = \frac{1-\nu^2}{2R}\frac{\partial}{\partial x}\nabla^2\Phi, \qquad v = \frac{1-\nu^2}{2R}\frac{\partial}{\partial y}\nabla^2\Phi, \qquad w = \frac{1-\nu}{2}\nabla^4\Phi \tag{2.16}$$

It follows from (2.16) that

$$u_y - v_x = 0, \qquad u_x + v_y = (1+\nu)\frac{w}{R} \tag{2.17}$$

Thus, in the case of a spherical shell, Expressions (1.9) and (1.10) cannot always be used; in fact, they can be used only when conditions (2.17) are satisfied. It will readily be seen that conditions (2.17) are sufficient for realizing solutions (1.9), (1.10). Let us consider now an example when conditions (2.17) are not satisfied. Suppose that

$$w = \frac{1}{a^3}(r^2 - a^2)^2, \qquad u = \frac{f(r)\,x}{a^3}, \qquad v = \frac{f(r)\,y}{a^3}$$

$$f(r) = \frac{1+\nu}{R}\left(\frac{r^4}{6} - \frac{a^2r^2}{2} + \frac{a^4}{3}\right), \quad r^2 = x^2 + y^2 \tag{2.18}$$

It will be seen that Expressions (2.18) satisfy (1.6), (1.7) and represent some symmetrical deformation of a shell fully restrained on the edge $r = a$. The corresponding loading can be found from (1.8). It can easily be shown that the first of conditions (2.17) is satisfied but the second is not, since

$$u_x + v_y - \frac{(1+\nu)\,w}{R} = -\frac{1+\nu}{3R}a \tag{2.19}$$

Thus, with the given conditions, (1.9), (1.10) are not possible.

In the case when (2.17) is satisfied and Φ_1, Φ_2 satisfy simultaneously (2.16)

$$\Phi_1 - \Phi_2 = \Phi^\circ + \alpha\,(x^2 + y^2) \tag{2.20}$$

where Φ° is some harmonic function and α is a constant.

3. Refer once again to (1.14), (1.15). Since the method of investigation here is analogous in every respect, we shall simply formulate the basic results, which reduce to the following.

a) Suppose that $k_x - k_y + ik_{xy} \neq 0$, i.e. that the shell is not spherical. In this case, for any sufficiently smooth functions related by (1.13), Expressions (1.15) are always possible.

b) If Φ_1, Φ_2 are two functions satisfying simultaneously (1.15), then

$$\Phi_1 - \Phi_2 = a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 \qquad (3$$

where the constants a_4, a_5, a_6 are arbitrary, and a_1, a_2, a_3 are related by the expression

$$a_1 k_x + a_2 k_y - 2k_{xy} a_3 = 0 \tag{3.2}$$

c) If the shell is spherical ($k_x = k_y = k$, $k_{xy} = 0$), Expressions (1.14), (1.15) are possible only if

$$w = Ehk \nabla^2 \Phi \tag{3.3}$$

Then, if Φ_1, Φ_2 satisfy simultaneously (1.15), $\Phi_1 - \Phi_2 = \Phi^\circ$, where Φ° is some harmonic function.

4. An examination of the expressions of Vlasov (1.20) to (1.22) leads to the following conclusions.

For any three functions related by (1.17), (1.18), Expressions (1.20) to (1.22) are always possible. Then, if Φ_1, Φ_2 satisfy simultaneously (1.20) to (1.22), we have

$$\Phi_1 - \Phi_2 = a_1 (x^3 + 3\nu x y^2) + a_2 [-3x^2 y + (2 + \nu) y^3] + P(x, y)$$

where P is an arbitrary second-order polynomial.

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